# Universal epimorphic equivalences for group localizations

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**Abstract**

Recent work by Bousfield shows the existence, for any map $\phi$, of a universal space that is killed by homotopical $\phi$-localization. Nullification with respect to this so-called universal $\phi$-acyclic space is related to $\phi$-localization in the same way as Quillen's plus construction is related to homological localization. Here we construct a universal $f$-acyclic group for any group homomorphism $f$. Moreover, we prove that there is a universal epimorphism $\mathcal{E}(f)$ that is inverted by $f$-localization. Although the kernel of the $\mathcal{E}(f)$-localization homomorphism coincides with that of the $f$-localization homomorphism, we show that localization with respect to $\mathcal{E}(f)$ has in general nicer properties than $f$-localization itself.

## 0 Introduction

In homotopy theory, Bousfield, Dror Farjoun and others have developed a theory of localization with respect to any given continuous map between topological spaces (cf. [3], [4], [11]). As pointed out by Casacuberta in [5], many of these results can be transferred to the category of groups. More precisely, to any group homomorphism $f$, it is associated a localization functor $L_f$ on the category of groups. The motivation for this paper comes from the fact that in many cases understanding homotopical localization requires this algebraic tool of localization with respect to a group homomorphism (cf. [6], [8]).

As in the case of spaces ([4]), we can consider the localization class of a group homomorphism $f$, denoted by $\langle f \rangle$, consisting of the collection of all group homomorphisms $g$

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yielding a localization functor $L_g$ that is naturally equivalent to $L_f$. The collection of all localization classes forms a small-complete lattice for an obvious partial order relation (cf. Section 1).

In homotopical localization, a fundamental role is played by the so-called universal $\varphi$-acyclic space $A(\varphi)$ for a continuous map $\varphi$ ([4]). The $A(\varphi)$-nullification (that is, localization with respect to the map $A(\varphi) \to *$) is related to localization with respect to the map $\varphi$ in the same way as Quillen's plus construction is related to Bousfield's homological localization. In fact, in terms of the above mentioned partial order relation, the localization class $\langle A(\varphi) \rangle$ is maximal among all nullification classes smaller than $\langle \varphi \rangle$. In Theorem 3.1 we prove that a similar result holds for groups in the context of localization with respect to a given homomorphism $f$. Moreover, we also prove that there exists a maximal class $\langle \mathcal{E}(f) \rangle$ which is smaller than $\langle f \rangle$ and where $\mathcal{E}(f)$ is an epimorphism.

We further show in Theorem 2.1 that $G \to L_f G$ is surjective for all groups $G$ if and only if the class $\langle f \rangle$ contains an epimorphism. We refer to these classes as epireflection classes. As explained in Section 2, to any epireflection class $\langle f \rangle$ we can associate a radical $R_f$ such that $L_f G \cong G/R_f G$ for all groups $G$. We characterize the nullifications as those epireflection classes for which the associated radical is idempotent (cf. Proposition 2.3). Using previous results on the preservation of exact sequences under localization with respect to an epimorphism (cf. Proposition 2.5), we further give characterizations of epireflection and nullification classes in terms of closure properties of the class of their local groups (cf. Theorem 2.6).

The construction given in Theorem 3.1 turned out to be very useful in [10], where it served to show that, under the assumption that all cardinals are nonmeasurable, there exists a localization functor of groups (resp. of spaces) which is not equivalent to $L_f$ for any group homomorphism (resp. map) $f$. Applications of our results in the case of epireflections and nullifications associated with varieties of groups and corresponding homotopical localizations, can be found in [9]. For a further, particularly interesting and motivating example we refer to [1], where the authors consider universal acyclic spaces for Bousfield's homological localization. In a forthcoming paper ([14]) we will treat universal epimorphisms for homological localization in relation to homology equivalences of spaces inducing an epimorphism on the fundamental group.

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# 1 The lattice of localization functors

Recall from [6] the definition of localization with respect to a given group homomorphism $f\colon A \to B$. A group $G$ is called *$f$-local* if the induced map of sets $f^*\colon \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ is a bijection. For every group $G$ there is a homomorphism $l_G\colon G \to L_f G$, which is initial among all homomorphisms from $G$ into $f$-local groups. $L_f$ is called the localization functor with respect to $f$. A group homomorphism $\phi\colon G \to H$ is called an *$f$-equivalence* if $L_f\phi\colon L_f G \to L_f H$ is an isomorphism. Further, a group $G$ is called *$f$-acyclic* if $L_f G = 1$. In the special case where $f$ is of the form $f\colon A \to 1$, the $f$-localization of a group $G$ is also denoted by $P_A G$ and it is called the *$A$-nullification* of $G$ (cf. [3], [6]). In this case the $f$-local groups are also called *$A$-null*.

We define the *localization class* of $f$, denoted by $\langle f \rangle$, as the collection of all homomorphisms $g$ such that $L_g$ is naturally equivalent to $L_f$. When $f$ is of the form $f\colon A \to 1$, the class of $f$ is simply denoted by $\langle A \rangle$ and is called the *nullification class* of $A$.

As in [4] we say that $\langle f \rangle \leq \langle g \rangle$ if and only if there is a natural transformation of localization functors $L_f \to L_g$, which is equivalent to every $g$-local group being $f$-local, or to every $f$-equivalence being a $g$-equivalence. This relation defines a partial order on the collection of localization functors with respect to group homomorphisms. Analogously to what happens for spaces (cf. [4]), each set of localization classes $\{\langle f_\alpha \rangle\}$ has a least upper bound, which we denote by $*\langle f_\alpha \rangle$. Note that a group $G$ is $f_\alpha$-local for all $\alpha$ if and only if $G$ is $(*f_\alpha)$-local, where now $*f_\alpha$ denotes the free product of all the homomorphisms $f_\alpha$. In other words, we have that $*\langle f_\alpha \rangle = \langle *f_\alpha \rangle$. Furthermore, parallelling Bousfield's argument in [4], it can also be shown that each set of localization classes has a greatest lower bound, so that the collection of localization functors with respect to group homomorphisms is a small-complete lattice.

Given a group homomorphism $f\colon A \to B$ and any group $G$, we can factor the localization homomorphism $l_G\colon G \to L_f G$ as the epimorphism $\pi_G\colon G \twoheadrightarrow \mathrm{im}\, l_G$ followed by the monomorphism $j_G\colon \mathrm{im}\, l_G \rightarrowtail L_f G$. It is then easy to check that

$$\langle f \rangle = \langle l_A \rangle * \langle l_B \rangle = \langle \pi_A \rangle * \langle \pi_B \rangle * \langle j_A \rangle * \langle j_B \rangle. \tag{1.1}$$

## 2 Classes of epimorphisms

As explained in Section 1, to any group homomorphism $f$ we can associate a localization functor $L_f$ on the category of groups. We next prove a claim made by Casacuberta in [6], and show that the converse equally holds. Observe that, for an epimorphism $f\colon A \to B$, a group $G$ is $f$-local if and only if the restriction of any homomorphism $A \to G$ to $\ker f$ is trivial.

**Theorem 2.1** *Let $f\colon A \to B$ be any group homomorphism. Then $G \to L_f G$ is surjective for all groups $G$ if and only if there exists an epimorphism $g$ such that $\langle f \rangle = \langle g \rangle$.*

PROOF. Assume that $f$ is an epimorphism. We construct a (possibly transfinite) ascending sequence $\{R_\alpha\}$ of normal subgroups of $G$, starting with $R_0 = \{1\}$. If $\alpha$ is a successor ordinal, take $R_\alpha / R_{\alpha-1}$ to be the subgroup of $G/R_{\alpha-1}$ generated by all elements of the form $\psi(a)$, where $a$ ranges over all elements in $\ker f$ and $\psi$ ranges over all homomorphisms $A \to G/R_{\alpha-1}$. For a limit ordinal $\lambda$ we set $R_\lambda = \lim_\alpha R_\alpha$, where the direct limit ranges over all ordinals $\alpha < \lambda$. Since each $R_\alpha$ is a subset of $G$, the system $\{R_\alpha\}$ stabilizes. If we denote the direct limit by $R_f G$, we conclude analogously as in [6, Theorem 3.2], that $L_f G \cong G/R_f G$. Conversely, if $l_G\colon G \to L_f G$ is surjective for all $G$, then $\langle f \rangle = \langle l_A \rangle * \langle l_B \rangle = \langle \pi_A \rangle * \langle \pi_B \rangle = \langle \pi_A * \pi_B \rangle$, by (1.1). □

Note that $R_f$ as constructed in the proof above is a radical on the category of groups, in the sense that it is a subfunctor of the identity functor and that $R_f(G/R_f G) = 1$ for all groups $G$. An alternative construction of this radical can be given as follows: $R_f G$ is the intersection of the kernels of all possible epimorphisms from $G$ onto $f$-local groups.

**Definition 2.2** The localization class $\langle f \rangle$ is called an *epireflection class* if the homomorphism $l_G\colon G \to L_f G$ is surjective for every group $G$.

Observe that, indeed, localization with respect to any epimorphism is an epireflection in the sense of [2].

We next characterize the nullification classes among the epireflection classes.

**Proposition 2.3** *Given a group homomorphism $f\colon A \to B$ such that $\langle f \rangle$ is an epireflection class, the following assertions are equivalent:*

(i) *$\langle f \rangle$ is a nullification class;*

(ii) *The radical* $R_f$ *is idempotent, i.e.* $R_f(R_fG) = R_fG$ *for all groups* $G$*;*

(iii) $R_fG$ *is* $f$*-acyclic for all groups* $G$*;*

(iv) $\langle f \rangle = \langle R_fA \rangle * \langle R_fB \rangle$*.*

PROOF. The fact that (i) implies (ii) was explained in [6, Theorem 3.2], while the implications (iv) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are direct. To see that (iii) implies (iv) note the following. If both $R_fA, R_fB$ are $f$-acyclic, then $\langle R_fA \rangle * \langle R_fB \rangle \leq \langle f \rangle$. Using (1.1), we infer that $\langle f \rangle = \langle l_A \rangle * \langle l_B \rangle \leq \langle \ker l_A \rangle * \langle \ker l_B \rangle = \langle R_fA \rangle * \langle R_fB \rangle$. □

**Example 2.4** For any free presentation $f\colon F \twoheadrightarrow F/K \cong G$ of a countable group $G$, where $F$ is a countably generated free group, the class of $f$-local groups is precisely the variety of groups satisfying the laws given by the words of $K$ ([13]), and $L_f$ is the projection onto this variety ([9]). Since the class of local groups of a nullification is closed under extensions (cf. Theorem 2.6), the class $\langle f \rangle$ is a nullification class if and only if the corresponding variety is either trivial (consisting only of the trivial group) or the variety of all groups. These are indeed the only varieties closed under extensions, as can easily be seen from Theorem 23.23 of [13].

In general, a localization functor $L_f$ does not behave well with respect to short exact sequences. However, for epireflections and nullifications, the following holds.

**Proposition 2.5** *Let* $f$ *be any group homomorphism and suppose given a short exact sequence of groups* $1 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 1$*, where* $Q$ *is* $f$*-local.*

(i) *If* $\langle f \rangle$ *is an epireflection class, then* $L_fN \to L_fG \to Q \to 1$ *is exact.*

(ii) *If* $\langle f \rangle$ *is a nullification class, then* $1 \to L_fN \to L_fG \to Q \to 1$ *is exact.*

PROOF. Part (i) follows easily by an appropriate diagram chase. Now, suppose that $\langle f \rangle$ is a nullification class, i.e. there exists a group $D$ such that $L_f$ and $P_D$ are naturally equivalent. We show that $P_D(i)$ is also injective. Let $K = \ker P_D(i)$. Since $K$ is the kernel of a homomorphism between $D$-null groups, $K$ is itself $D$-null (cf. [11]). But $K$ is also a quotient of $R_DG = \ker(G \to P_DG)$, which is a $D$-acyclic group by Proposition 2.3, so that $K$ itself is $D$-acyclic (cf. [11]). Since $K$ is both $D$-acyclic and $D$-null, we conclude that $K = 1$. □

As explained in [7], the arguments of Hilton in [12] to establish localization of relative nilpotent groups, can be extended to yield a relative $f$-localization for any group homomorphism $f$. This means that any short exact sequence of groups $1 \to N \to G \to Q \to 1$ can be embedded into a diagram

$$\begin{array}{ccccccccc} 1 & \to & N & \to & G & \to & Q & \to & 1 \\ & & \downarrow l_N & & \downarrow l & & \downarrow \mathrm{id} & & \\ 1 & \to & L_f N & \to & LG & \to & Q & \to & 1, \end{array}$$

where $l$ is an $f$-equivalence (compare with fibrewise localization in homotopy theory ([11])). This relative $f$-localization enables us to prove part (ii) of the following theorem. (Compare also part (i) with [2, Proposition 3.6.2].)

**Theorem 2.6** *Let $f$ be any group homomorphism. Then*

(i) *$\langle f \rangle$ is an epireflection class if and only if the class of $f$-local groups is closed under subgroups;*

(ii) *$\langle f \rangle$ is a nullification class if and only if the class of $f$-local groups is closed under subgroups and extensions.*

Proof. Suppose that the class of $f$-local groups is closed under subgroups. Let $G$ be any group and factor $l_G \colon G \to L_f G$ as $\pi_G \colon G \twoheadrightarrow \operatorname{im} l_G$ followed by $j_G \colon \operatorname{im} l_G \rightarrowtail L_f G$. By hypothesis, $\operatorname{im} l_G$ is $f$-local as subgroup of an $f$-local group. However, by the universal property of the localization, this implies that $\operatorname{im} l_G \cong L_f G$. Hence, $l_G$ is surjective. The other implication being obvious, this proves part (i).

By part (i) and by Proposition 2.5, the class of $f$-local groups of a nullification is closed under subgroups and extensions. Conversely, if the class of $f$-local groups is closed under subgroups, we may assume by (i) that $f$ is an epimorphism. Hence, there is a radical $R_f$ associated to $L_f$ such that $L_f G \cong G/R_f G$ for any group $G$. Using relative $f$-localization, we obtain a diagram

$$\begin{array}{ccccccccc} 1 & \to & R_f G & \to & G & \to & L_f G & \to & 1 \\ & & \downarrow & & \downarrow l & & \downarrow \mathrm{id} & & \\ 1 & \to & L_f R_f G & \to & LG & \to & L_f G & \to & 1, \end{array}$$

where $l$ is an $f$-equivalence. By assumption, $LG$ is $f$-local since it is an extension of $f$-local groups. Hence, $LG \cong L_f G$, from which we infer that $L_f R_f G = 1$. By Proposition 2.3, we thus conclude that $\langle f \rangle$ is a nullification class. □

## 3 Universal acyclic groups and universal epimorphic equivalences

Parallelling Bousfield's arguments in [4], we here prove that, given a group homomorphism $f$, there exists a maximal nullification class $\langle A(f)\rangle$ that is smaller than $\langle f\rangle$. Accordingly we call $A(f)$ a *universal $f$-acyclic group*, since it has the following universal property: a group $G$ is $f$-acyclic if and only if $G$ is $A(f)$-acyclic. Moreover, we also prove that, given a localization class $\langle f\rangle$, there exists a maximal class $\langle \mathcal{E}(f)\rangle$ which is smaller than $\langle f\rangle$ and where $\mathcal{E}(f)$ is an epimorphism. We call $\mathcal{E}(f)$ a *universal epimorphic $f$-equivalence*, since an epimorphism $g$ is an $f$-equivalence if and only if it is an $\mathcal{E}(f)$-equivalence.

**Theorem 3.1** *Let $f\colon A \to B$ be any group homomorphism. Then there exist a maximal nullification class $\langle A(f)\rangle$ and a maximal epireflection class $\langle \mathcal{E}(f)\rangle$ such that*

$$\langle A(f)\rangle \leq \langle \mathcal{E}(f)\rangle \leq \langle f\rangle.$$

Proof. The idea is to parallel the argument of the existence of a universal $\varphi$-acyclic space given by Bousfield in [4]. Define a functor $T_f$, from the category of group homomorphisms to the category of pointed sets, given by $T_f(g) = \ker L_f(g) \vee (L_f H/\mathrm{im}\, L_f(g))$ for a group homomorphism $g\colon G \to H$, where $\vee$ denotes the pointed union and / denotes the quotient as pointed sets. This functor $T_f$ is coherent and measures whether a homomorphism $g$ is an $f$-equivalence (cf. [4, Theorem 3.5]). By an obvious analogue to [4, Lemma 3.2], there exists an infinite cardinal $2^d$ such that each $f$-acyclic group is the colimit of a directed system of $f$-acyclic subgroups of cardinality $\leq 2^d$. Define then $A(f)$ to be the free product of a set of representatives of isomorphism classes of $f$-acyclic groups of cardinality $\leq 2^d$. Likewise every epimorphic $f$-equivalence $g\colon G \twoheadrightarrow H$ is the colimit of a directed system of sub-$f$-equivalences $\{\varphi_i\colon G_i \to H_i\}$ of cardinality $\leq 2^d$. Since $g$ is clearly also the colimit of the directed system $\{\tilde{\varphi}_i\colon G_i \twoheadrightarrow \mathrm{im}\,\varphi_i\}$, we take $\mathcal{E}(f)$ to be the free product of a set of representatives of isomorphism classes of epimorphic $f$-equivalences of cardinality at most $2^d$. □

Observe that the above theorem says that every localization $L_f$ has a best possible approximation by a nullification and by an epireflection. Indeed, for every group $G$, we have natural homomorphisms $G \to P_{A(f)}G \to L_{\mathcal{E}(f)}G \to L_f G$. Furthermore, the homomorphism $G \to L_{\mathcal{E}(f)}G$ is terminal among all epimorphic $f$-equivalences going out of $G$, while $G \to P_{A(f)}G$ is terminal among all epimorphic $f$-equivalences going out of $G$ and having an $f$-acyclic kernel.

As we next prove, the radical associated to a universal epimorphic $f$-equivalence $\mathcal{E}(f)$ (in the sense explained in Section 2) coincides with the kernel of the $f$-localization homomorphism.

**Theorem 3.2** *Let $f$ be any group homomorphism and denote by $\mathcal{E}(f)$ a universal epimorphic $f$-equivalence. Then*

$$L_{\mathcal{E}(f)}G \cong G/\ker l_G$$

*for all groups $G$, where $l_G\colon G \to L_fG$ denotes the localization homomorphism.*

PROOF. We show that $\pi_G\colon G \twoheadrightarrow \operatorname{im} l_G$ is the $\mathcal{E}(f)$-localization of $G$. ¿From (1.1) we know that $\pi_G$ is an epimorphic $f$-equivalence, and hence an $\mathcal{E}(f)$-equivalence. Moreover, $L_fG$ is an $\mathcal{E}(f)$-local group, which implies by Theorem 2.6 (i) that $\operatorname{im} l_G$ is $\mathcal{E}(f)$-local. □

The existence of a universal $f$-acyclic group was of crucial importance in [10], where it enabled to prove that, under the assumption that all cardinals are nonmeasurable, there exits both a localization functor of groups and of spaces which are not of the form $L_g$ for any group homomorphism or map $g$. For $H\mathbb{Z}$-localization of groups (i.e., localization with respect to integral homology), which is equivalent to $L_f$ for some homomorphism $f$ between free groups ([11]), an explicit construction of a locally free, universal $f$-acyclic group $A(f)$ was given by Berrick and Casacuberta in [1]. In this case homotopical nullification with respect to $K(A(f),1)$ was shown to be equivalent to Quillen's plus construction. In [14] we will consider universal epimorphic $H\mathbb{Z}$-equivalences of groups and relate them to homology equivalences of spaces inducing an epimorphism on the fundamental group.

# References


[1] J. Berrick and C. Casacuberta, *A universal space for plus constructions*, Topology (to appear).

[2] F. Borceux, *Handbook of categorical algebra I*, Cambridge University Press, Cambridge (1994).

[3] A. K. Bousfield, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. 7 (1994), 831–873.

[4] A. K. Bousfield, *Homotopical localizations of spaces*, Amer. J. Math. 119 (1997), 1321–1354.

[5] C. Casacuberta, *Recent advances in unstable localization*, in: The Hilton Symposium 1993; Topics in Topology and Group Theory, CRM Proceedings and Lecture Notes 6, Amer. Math. Soc., Providence (1994), 1–22.

[6] C. Casacuberta, *Anderson localization from a modern point of view*, in: The Čech Centennial, Contemporary Math. 181, Amer. Math. Soc., Providence (1995), 35–44.

[7] C. Casacuberta and A. Descheemaeker, *Relative localizations of groups*, in preparation.

[8] C. Casacuberta, J. L. Rodríguez, and Jin-Yen Tai, *Localizations of abelian Eilenberg–Mac Lane spaces of finite type*, preprint (1997).

[9] C. Casacuberta, J. L. Rodríguez, and D. Scevenels, *Singly generated radicals associated with varieties*, in: Proceedings of Groups St Andrews 1997 in Bath, London Math. Soc. Lecture Note Series, Cambridge University Press, to appear.

[10] C. Casacuberta, D. Scevenels and J. H. Smith, *Implications of large-cardinal principles in homotopical localization*, preprint (1998).

[11] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Math. 1622, Springer-Verlag, Berlin Heidelberg New York (1996).

[12] P. Hilton, *Relative nilpotent groups*, in: Categorical Aspects of Topology and Analysis, Lecture Notes in Math. 915, Springer-Verlag, Berlin Heidelberg New York (1982), 136–147.

[13] H. Neumann, *Varieties of groups*, Springer-Verlag, Berlin Heidelberg New York (1967).

[14] J. L. Rodríguez and D. Scevenels, *Homology equivalences inducing an epimorphism on the fundamental group*, in preparation.

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